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Triangle UD integrals in the position space

Igor Kondrashuk

Departamento de Ciencias Básicas, Universidad del Bío-Bío,
Campus Fernando May, Casilla 447, Chillán, Chile
E-mail: igor.kondrashuk@ubiobio.cl

Anatoly Kotikov

Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research,
Dubna, Russia, and
II. Institut fur Theoretische Physik, Universitat Hamburg,
Luruper Chaussee 149, 22761 Hamburg, Germany
E-mail: kotikov@mail.desy.de

Abstract: We investigate triangle UD ladder integrals in the position space. The investigation is necessary to find an all-order in loop solution for an auxiliary *Lcc* correlator in Wess-Zumino-Landau gauge of the maximally supersymmetric Yang-Mills theory and to present correlators of dressed mean gluons in terms of it in all loops. We show that triangle UD ladder diagrams in the position space can be expressed in terms of the same UD functions $\Phi^{(L)}$, in terms of which they were represented in the momentum space, for an arbitrary number of rungs.

Keywords: Supersymmetric gauge theory, Extended Supersymmetry, BRST Quantization, BRST Symmetry.

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As has been shown in refs. [1]–[12], Slavnov-Taylor (ST) identity predicts that the correlators of dressed mean fields for $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in *the position space* can be represented in terms of Usyukina-Davydychev (UD) integrals (at least at the two loop planar level). Indeed, at that level the auxiliary Lcc correlator in the position space in Wess-Zumino-Landau gauge of the maximally supersymmetric Yang-Mills theory is a function of Davydychev integral $J(1,1,1)$ [4 − 6] which is the first integral in the chain of UD integrals [13 − 15]. By using ST identity, we can express all the correlators in terms of this correlator in that theory. Furthermore, using the method of ref. [6], one can expect that at the higher loop orders in the position space the triangle UD ladder integral contributions to that auxiliary correlator will survive only. Strictly speaking, the powers of d'Alambertian applied to the L-field vertex of the triangle ladder will contribute only. In this paper we show that such constructions are the UD functions of the spacetime intervals. Conformal invariance of the effective action of dressed mean fields in the position space, suggested in refs. [1 − 3], corresponds to the property of conformal invariance of the UD functions in the position space.

The UD integrals correspond to the momentum representation of three-point ladder diagrams (triangle ladders) and four-point ladder diagrams and were defined and calculated in refs. [14, 15] in *the momentum space*, and the result can be written in terms of the UD functions $\Phi^{(L)}$ of conformally invariant ratios of momenta.[1] In the momentum space it was shown that the UD functions are the only contributions (at least up to three loops) to off-shell four-point correlator of gluons that corresponds to four gluon amplitude [18, 19]. The conformal invariance of UD functions was used in the momentum space to calculate four-point amplitude and to classify all possible contributions to it [20, 21]. Later, the conformal symmetry in the momentum space appeared on the string side in the Alday-Maldacena approach [22] in the limit of strong coupling.

In this paper we use two things known from the literature. These are the iterative definition of the UD functions, that is eq. (23) of ref. [14], and the dual graphical representation for four-point momentum UD integrals in the form of "diamonds" [23, 18]. Before starting the demonstration, we outline some basic points of it. In ref. [7] we have proved the identity[2]

$$\int d^4y\, d^4z \frac{1}{[2y][1y][3z][yz][2z]} = \frac{1}{[31]}\Phi^{(2)}\left(\frac{[12]}{[31]}, \frac{[23]}{[31]}\right) \tag{1}$$

by conformal transformation of the integrand. The l.h.s. of this relation corresponds to the l.h.s. of the line (a) of figure 1. In this paper we assume the notation of ref. [4], where $[Ny] = (x_N - y)^2$ and analogously for $[Nz]$ and $[yz]$, that is, $N = 1, 2, 3$ stands for $x_N = x_1, x_2, x_3$, respectively, throughout all the paper. In the momentum space, eq. (1) might be understood as a relation between the momentum integrals that correspond to the l.h.s. and the r.h.s. of figure 2 of ref. [24] which was derived by using the trick of integration

[1]In the position space Feynman diagrams contain integrations over coordinates of internal vertices. Integration over internal vertices appears in dual representation of the momentum diagrams too [16, 17]

[2]Our definition for UD functions is $\Phi^{(L)}_{\mathrm{New}} = (\pi^2)^L \Phi^{(L)}_{\mathrm{Old}}$, where $\Phi^{(L)}_{\mathrm{New}}$ is $\Phi^{(L)}$ of this paper, and $\Phi^{(L)}_{\mathrm{Old}}$ is the original UD function $\Phi^{(L)}$ of refs. [14, 15].

by parts. However, in ref. [24] the position space picture was not analysed and it was not shown that Fourier transform of the second UD integral is the same integral [7]. Here we demonstrate the validity of a similar property for any UD function.

The line (b) of figure 1 is figure 5 of ref. [7]. It corresponds to the equation

$$\partial^2_{(2)} \int Dy\ Dz \frac{1}{[2y][1y][3z][yz][2z]} = -\frac{4[31]}{[12][23]}\ J(1,1,1). \tag{2}$$

This equation has been generated by the computer program of ref. [6] by making use of formulas of refs. [4, 5] derived by Gegenbauer polynomial technique. Without modifications of the integral measure made in ref. [4] this equation is[3]

$$\partial^2_{(2)} \int d^4y\ d^4z \frac{1}{[2y][1y][3z][yz][2z]} = -\frac{4\pi^2[31]}{[12][23]}\ J(1,1,1). \tag{3}$$

In ref. [25] this equation was obtained by direct differentiation of the second UD integral. It can be demonstrated in several ways. For example, the direct differentiation of the r.h.s. of eq. (1) using the eq. (23) of ref. [14] and eq. (12) of ref. [15] produces

$$\begin{aligned}
\partial^2_{(2)} \frac{1}{[31]}\Phi^{(2)}\left(\frac{[12]}{[31]},\frac{[23]}{[31]}\right) &= [31]\partial^2_{(2)}\ \frac{1}{[31]^2}\Phi^{(2)}\left(\frac{[12]}{[31]},\frac{[23]}{[31]}\right) = \ [31]\partial^2_{(2)}C^{(2)}([12],[23],[31]) \\
&= [31]\partial^2_{(2)} \int\ d^4y\ \frac{C^{(1)}([(12)+y],[(23)-y],[31])}{[(12)+y]\ [(23)-y]\ [y]} \\
&= [31]\partial^2_{(2)} \int\ d^4y\ \frac{C^{(1)}([(12)-y],[(32)-y],[31])}{[(12)-y]\ [(32)-y]\ [y]} \\
&= \ [31]\partial^2_{(2)} \int\ d^4y\ \frac{C^{(1)}([1y],[3y],[31])}{[1y]\ [2y]\ [3y]} \\
&= -4\pi^2[31] \int\ d^4y\ \delta(2y)\frac{C^{(1)}([1y],[3y],[31])}{[1y]\ [3y]} \\
&= -4\pi^2[31]\frac{C^{(1)}([12],[23],[31])}{[12]\ [23]} \\
&= -\frac{4\pi^2}{[12][23]}\ \Phi^{(1)}\left(\frac{[12]}{[31]},\frac{[23]}{[31]}\right).
\end{aligned} \tag{4}$$

On the other hand, by using conformal transformation the r.h.s. of eq. (3) and eq. (4) can be related. Three-point UD functions can be transformed to four-point UD functions due to Jacobian of conformal transformation, since under this transformation each three-point internal vertex transforms to four-point internal vertex with a new leg growing from the internal vertex to the point 0 which is the inition of the reference system. The conformal substitution for each vector of the integrand (including the external vectors) is

$$y_\mu = \frac{y'_\mu}{y'^2}, \qquad z_\mu = \frac{z'_\mu}{z'^2}, \tag{5}$$

[3] All internal vertices of the diagrams in this paper correspond to the standard four-dimensional integral measure.

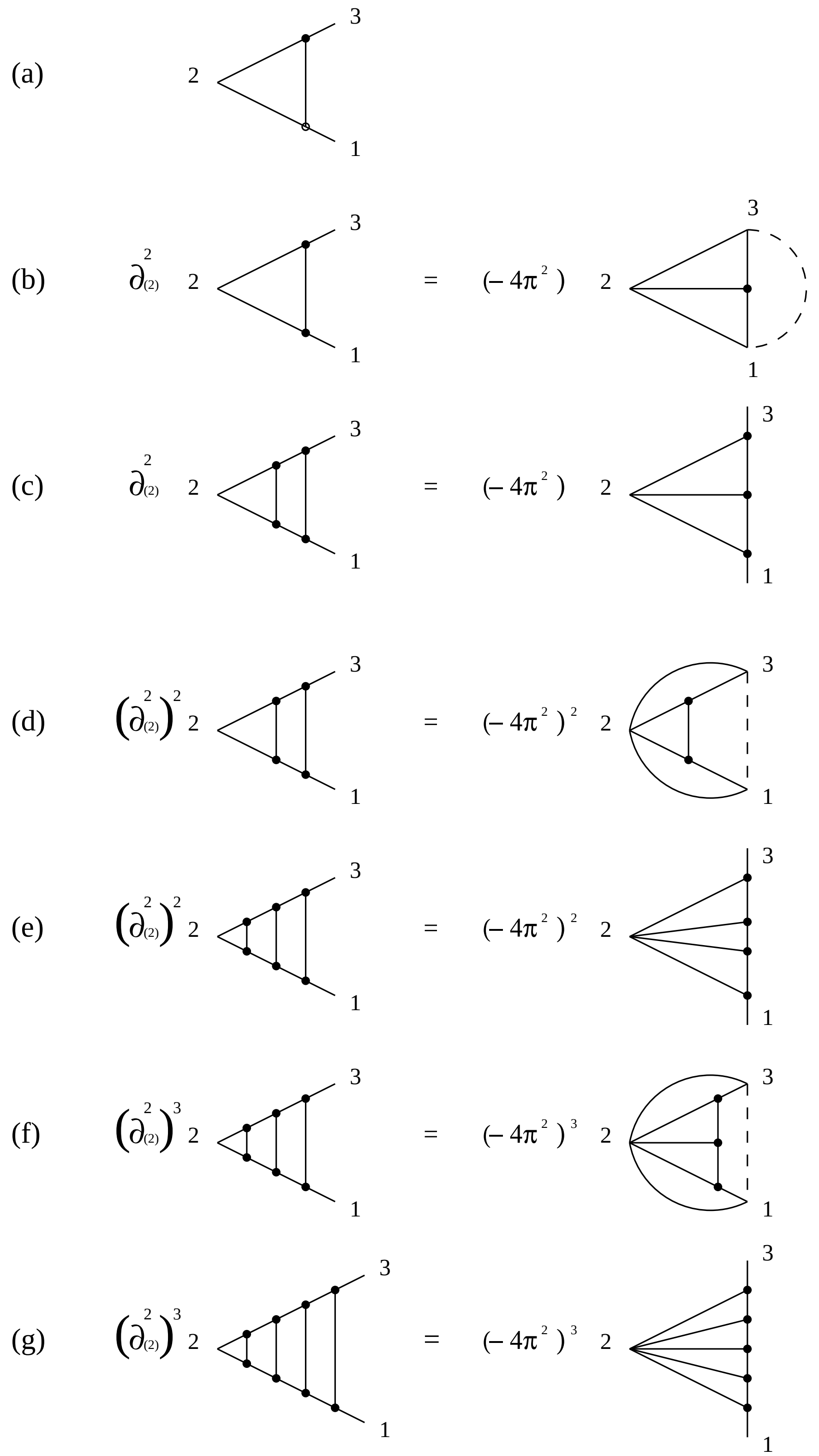

Figure 1: Chain of transformations

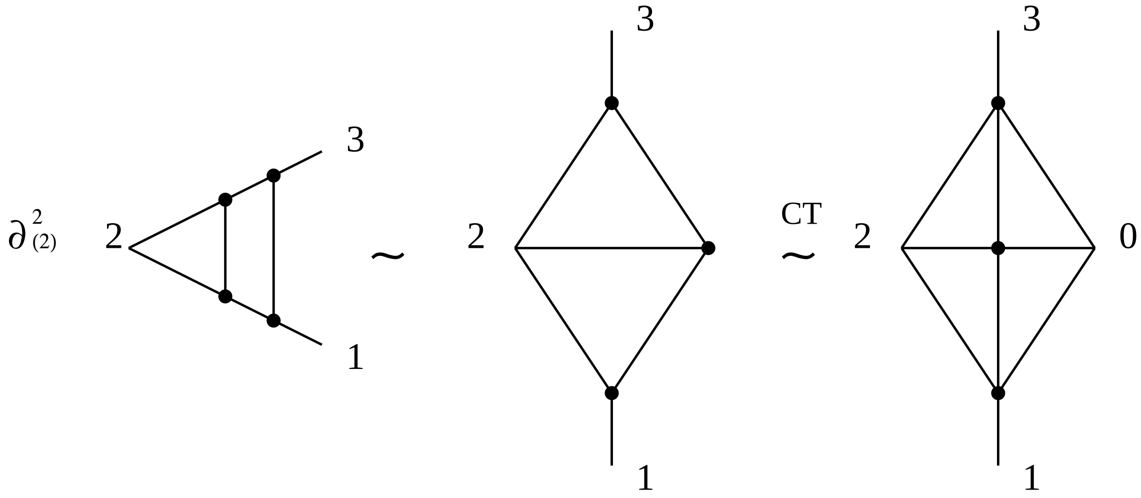

Figure 2: Transformation of two rungs diagram

and in the simplest case of the first UD function we have

$$
\begin{aligned}
J(1,1,1) = \int d^4y \, \frac{1}{[1y][2y][3y]} &= [1'][2'][3'] \int d^4y' \frac{1}{[1'y'][2'y'][3'y'][y']} \\
&= [1'][2'][3'] \frac{1}{[3'1'][2']} \Phi^{(1)} \left(\frac{[1'2'][3']}{[3'1'][2']}, \frac{[1'][2'3']}{[3'1'][2']} \right) \\
&= \frac{[1'][3']}{[3'1']} \Phi^{(1)} \left(\frac{[1'2'][3']}{[3'1'][2']}, \frac{[1'][2'3']}{[3'1'][2']} \right) = \frac{1}{[31]} \Phi^{(1)} \left(\frac{[12]}{[31]}, \frac{[23]}{[31]} \right).
\end{aligned}
$$

The line (c) is the direct use of the line (b) and, as it can be proved by the sequence of transformations depicted in figure 2, its r.h.s. is proportional to $\Phi^{(3)}$, indeed

$$
\begin{aligned}
&\int d^4y \; d^4z \; d^4u \frac{1}{[2y][2u][2z][3z][1y][uz][uy]} \\
&\quad = [2']^3[3'][1'] \int d^4y' \; d^4z' \; d^4u' \frac{1}{[2'y'][2'u'][2'z'][3'z'][1'y'][u'z'][u'y'][y'][z'][u']} \\
&\quad = [2']^3[3'][1'] \frac{1}{[3'1'][2']^3} \Phi^{(3)} \left(\frac{[1'2'][3']}{[3'1'][2']}, \frac{[1'][2'3']}{[3'1'][2']} \right) \\
&\quad = \frac{[3'][1']}{[3'1']} \Phi^{(3)} \left(\frac{[1'2'][3']}{[3'1'][2']}, \frac{[1'][2'3']}{[3'1'][2']} \right) = \frac{1}{[31]} \Phi^{(3)} \left(\frac{[12]}{[31]}, \frac{[23]}{[31]} \right).
\end{aligned} \tag{6}
$$

The factor $-4\pi^2$ on the r.h.s. of the line (c) came from eq. (4).

The line (d) can be obtained by the direct differentiation of the previous result,

$$
\begin{aligned}
&\partial^2_{(2)} \frac{1}{[31]} \Phi^{(3)} \left(\frac{[12]}{[31]}, \frac{[23]}{[31]} \right) = \\
&\qquad = [31]^2 \partial^2_{(2)} \; \frac{1}{[31]^3} \Phi^{(3)} \left(\frac{[12]}{[31]}, \frac{[23]}{[31]} \right) = \; [31]^2 \partial^2_{(2)} C^{(3)}([12],[23],[31]) \\
&\qquad = [31]^2 \partial^2_{(2)} \int \; d^4y \; \frac{C^{(2)}([(12)+y],[(23)-y],[31])}{[(12)+y] \; [(23)-y] \; [y]}
\end{aligned}
$$

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$$
\begin{aligned}
&= [31]^2 \partial^2_{(2)} \int \, d^4y \; \frac{C^{(2)}([(12)-y],[(32)-y],[31])}{[(12)-y] \; [(32)-y] \; [y]} \\
&= [31]^2 \partial^2_{(2)} \int \, d^4y \; \frac{C^{(2)}([1y],[3y],[31])}{[1y] \; [2y] \; [3y]} = \\
&= -4\pi^2 [31]^2 \int \, d^4y \; \delta(2y) \frac{C^{(2)}([1y],[3y],[31])}{[1y] \; [3y]} = -4\pi^2 [31]^2 \frac{C^{(2)}([12],[23],[31])}{[12] \; [23]} \\
&= -\frac{4\pi^2}{[12][23]} \; \Phi^{(2)} \left(\frac{[12]}{[31]}, \frac{[23]}{[31]} \right). \qquad (7)
\end{aligned}
$$

Using this and taking into account the line (a) and the result for it represented by eq. (1) we have obtained the r.h.s. of line (d).

The line (e) is the direct use of the line (d). The r.h.s. of the line (e) is proportional to

$$
\frac{1}{[31]} \Phi^{(4)} \left(\frac{[12]}{[31]}, \frac{[23]}{[31]} \right). \qquad (8)
$$

The proof of this statement repeats proof (6), the only difference is that instead of three internal vertices between the points 1 and 3 on the r.h.s. of the line (c), we have four internal vertices for the r.h.s. of the line (e).

The line (f) is the repetition of the trick of eq. (4) and eq. (7) with the lines (b) and (d). Indeed, applying d'Alambertian to eq. (8), we obtain

$$
\partial^2_{(2)} \frac{1}{[31]} \Phi^{(4)} \left(\frac{[12]}{[31]}, \frac{[23]}{[31]} \right) = -\frac{4\pi^2}{[12][23]} \; \Phi^{(3)} \left(\frac{[12]}{[31]}, \frac{[23]}{[31]} \right). \qquad (9)
$$

Using this and taking into account the line (c) and the result for its r.h.s. represented by eq. (6) we have obtained the r.h.s. of the line (f).

The line (g) is the direct use of line (f). Repeating the proof of eq. (6) we obtain that the r.h.s. of the line (g) is proportional to

$$
\frac{1}{[31]} \Phi^{(5)} \left(\frac{[12]}{[31]}, \frac{[23]}{[31]} \right).
$$

We can proceed this chain of constructions to an arbitrary number of rungs, and analysing the previous results and figure 1, we obtain for n-rungs triangle UD ladder diagram $T_n([12],[23],[31])$ in the position space the following relations

$$
\begin{aligned}
\left(\partial^2_{(2)}\right)^{n-1} \; T_n([12],[23],[31]) &= \frac{(-4\pi^2)^{n-1}}{[31]} \Phi^{(n+1)} \left(\frac{[12]}{[31]}, \frac{[23]}{[31]} \right), \\
\left(\partial^2_{(2)}\right)^{n} \; T_n([12],[23],[31]) &= \frac{(-4\pi^2)^{n}}{[12][23]} \Phi^{(n)} \left(\frac{[12]}{[31]}, \frac{[23]}{[31]} \right), \\
\left(\partial^2_{(2)}\right)^{n} \; T_{n+1}([12],[23],[31]) &= \frac{(-4\pi^2)^{n}}{[31]} \Phi^{(n+2)} \left(\frac{[12]}{[31]}, \frac{[23]}{[31]} \right).
\end{aligned}
$$

These relations show that the auxiliary Lcc correlator in the position space in the maximally supersymmetric Yang-Mills theory can be represented in all loops in terms of the UD functions. Indeed, according to the technique developed in ref. [6], any contribution

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to the correlator can be expressed in terms of powers of d'Alambertian applied to a leg of scalar integrals. Using the graphical identity of refs. [7, 24] any four-point internal vertex of those scalar integrals can be presented in terms of three-point internal vertices. Other distributions of d'Alambertian produce Dirac delta-functions which will shrink one of the integration in the position space. To find the coefficients in front of the UD functions we need to solve Bethe-Salpeter equation for this double-ghost correlator. Furthermore, in terms of this correlator all the correlators of dressed mean gluons can be expressed by using Slavnov-Taylor identity. Thus, we can conclude that the correlators of dressed mean fields in that theory which are off-shell correlators in the position space are very complicated combinations of the three-point UD functions of space-time intervals.

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